Summing the Geometric Series

In lecture we saw a geometric argument that $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = 2$. By answering the questions below, we complete an algebraic proof that this is true.

We start by proving by induction that:

$$S_N = \sum_{n=0}^{N} \frac{1}{2^n} = \frac{2^{N+1} - 1}{2^N}.$$

Finally we show that $\lim_{N \to \infty} S_N = 2$.

a) (Base case) Prove that $S_0 = \frac{2^1 - 1}{2^0} = 1$.

b) (Inductive hypothesis and inductive step) Assume that:

$$S_{N-1} = \frac{2^{(N-1)+1} - 1}{2^{N-1}} = \frac{2^N - 1}{2^{N-1}}.$$

Add $\frac{1}{2^N}$ to both sides to prove that:

$$S_N = \frac{2^{N+1} - 1}{2^N}.$$

This completes the inductive proof.

c) Show that if $S_N = \frac{2^{N+1} - 1}{2^N}$, then $\lim_{N \to \infty} S_N = 2$.

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b) (Inductive hypothesis and inductive step) Assume that:

$$S_{N-1}=\frac{2^{(N-1)+1}-1}{2^{N-1}}=\frac{2^N-1}{2^{N-1}}.$$

Add $\frac{1}{2^N}$ to both sides to prove that:

$$S_N=\frac{2^{N+1}-1}{2^N}.$$

This completes the inductive proof.

c) Show that if $S_N=\frac{2^{N+1}-1}{2^N}$, then $\lim_{N\to\infty} S_N=2$.

a) $S_N=\sum_{n=0}^{N}\frac{1}{2^n} \Rightarrow S_0=\frac{1}{2^0}=1 \Rightarrow S_N=\frac{2^{N+1}-1}{2^N}=1$ for $N=0$.

b) Assume that $S_{N-1}=\frac{2^{(N-1)+1}-1}{2^{N-1}}$ holds.

$$S_{N-1}+\frac{1}{2^N}=\frac{2^{(N-1)+1}-1}{2^{N-1}}+\frac{1}{2^N}$$

$$\Rightarrow \quad S_N=\frac{2^N-1}{2^{N-1}}\left(\frac{2}{2}\right)+\frac{1}{2^N}=\frac{2^{N+1}-2}{2^N}+\frac{1}{2^N}=\frac{2^{N+1}-1}{2^N}.$$

$\therefore$ S_N is true and hence S_{N-1} is also true.

Hence, S_n is true for all $n\in\mathbb{N}$, $n\geq 0$.

c)

$$\lim_{N\to\infty} S_N$$

$$= \lim_{N\to\infty} \sum_{n=0}^{N} \frac{1}{2^n}$$

$$= \lim_{N\to\infty} \frac{2^{N+1}-1}{2^N}$$

$$= \lim_{N\to\infty} \frac{2^{N+1}}{2^N} - \frac{1}{2^N}$$

$$= \lim_{N\to\infty} 2 - \frac{1}{2^N}$$

$$= 2 - 0$$

$$= 2$$